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THE EFFECT OF WALL CONDUCTION ON HEAT TRANSFER TO A SLUG FLOW

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Abstract—The problem of heat transfer to a slug flow past a flat plate of finite dimensions is considered. Heat is introduced to the system either by means of a known flux at the outside surface of the plate or by means of a distributed heat source in the plate. Two dimensional conduction occurs in the plate. The temperature distribution in the plate and fluid is found in terms of a series expansion.

NOMENCLATURE

a,	plate thickness;
A_n, B_n,	Fourier coefficients;
b,	$= a/L$;
k_f, k_s,	solid and fluid conductivities;
K,	$= k_s/k_f$;
L,	plate length;
q_w,	heat introduced at plate surface;
q_s,	heat generation in solid;
Q_w,	$= q_w L/k_s T_{f0}$;
Q_s,	$= q_s L^2/k_s T_{f0}$;
s,	Laplace transform variable;
T_f, T_s,	fluid and solid temperatures;
T_{f0},	inlet fluid temperature;
v_x,	fluid velocity;
V,	$= v_x L/\alpha$;
x,	direction along plate;
X,	$= x/L$;
y,	direction normal to plate;
Y,	$= y/L$;
α,	fluid thermal diffusivity;
λ_n,	eigenvalue;
θ_s,	$= (T_s - T_{f0})/T_{f0}$;
θ_f,	$= (T_f - T_{f0})/T_{f0}$.

INTRODUCTION

In most analyses of heat transfer to forced convective flow, the temperature, the heat flux, or a combination of the two is given on the boundaries; the temperature distribution in the fluid and (or) the heat flux into the fluid are then determined. In some cases, however, the temperature or flux conditions at the bounding surface cannot be known *a priori*. In these cases, it is necessary to solve the equation of energy transport for the fluid simultaneously with the energy equation for the bounding solid. The equations are coupled by appropriate boundary conditions at the fluid–solid interface.

Heat transfer to slug flow between flat plates and in a circular tube has been studied by Chu and Bankoff [1]. A step temperature change was imposed on the outside surface and the heat flux and temperature on the inside surface found by means of a Fourier transform. Conduction occurs in the infinite length walls in two directions, i.e. parallel to and normal to the direction of flow. Numerical results show that the wall conduction smooths out the region of high heat flux which would occur if the wall had negligible thickness. Perelman [2] considered the problem of slip flow past a semi-infinite (semi-infinite both parallel to and normal to the direction of flow) solid having distributed heat sources. An analytical solution was obtained for the integral equation arising from coupling the Laplace transform solution for the fluid and the Fourier transform solution for the solid. No numerical results were given. Perelman also considered the more complex problem of transport to a laminar boundary layer. Some studies have also been made of the effect of one dimensional wall conduction in smoothing out peripheral temperature gradients in circular tubes [3] and in rectangular channels [4].

In this work, fluid with a constant velocity profile flowing past a finite length plate will be considered. The system is shown in Fig. 1. The

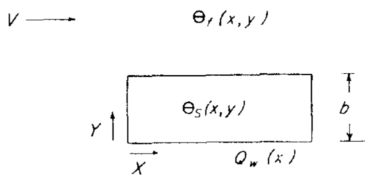

FIG. 1. Plate and fluid.

fluid approaches at a known temperature, and heat is introduced by means of a known heat flux at the bottom of the plate or by distributed heat sources in the solid. Conduction occurs in the plate in both the longitudinal and transverse directions; heat flows in the fluid by convection in the direction of flow and by conduction away from the plate. Expressions for the temperature distributions in the solid and in the fluid are found and the effect of two dimensional wall conduction on heat transfer to a slug flow is determined.

ANALYSIS

For slug flow, constant fluid and solid properties, and negligible fluid conduction in the direction of motion, the dimensionless energy equations for the fluid and solid respectively are

$$V\frac{\partial \theta_f}{\partial X} = \frac{\partial^2 \theta_f}{\partial Y^2} \qquad (1)$$

$$\frac{\partial^2 \theta_s}{\partial X^2} + \frac{\partial^2 \theta_s}{\partial Y^2} = -Q_s \qquad (2)$$

with boundary conditions

$$X = 0 \qquad \theta_f = 0 \qquad b < Y < \infty$$

$$Y = \infty \qquad \theta_f = 0$$

$$\left.\begin{matrix} X = 0 \\ X = 1 \end{matrix}\right\} \qquad \frac{\partial \theta_s}{\partial X} = 0 \qquad 0 < Y < b$$

$$Y = 0 \qquad \frac{\partial \theta_s}{\partial Y} = -Q_w$$

$$Y = b \qquad \theta_s = \theta_f$$

$$Y = b \qquad K\frac{\partial \theta_s}{\partial Y} = \frac{\partial \theta_f}{\partial Y}$$

The last two conditions insure continuity of temperature and heat flux at the interface between the solid and fluid.

Consider first the case in which $Q_w(X)$, the heat flux at $Y = 0$, is an arbitrary function of X but in which Q_s, the heat generated within the solid, is zero. Under these conditions, the solution of equation (2) satisfying the boundary conditions at $X = 0$, $X = 1$, and $Y = 0$ is

$$\theta_s = A_0 Y + B_0 + \sum_{n=1}^{\infty} \cos \lambda_n X(A_n \sinh \lambda_n Y + B_n \cosh \lambda_n Y) \qquad (3)$$

where $\lambda_n = n\pi$

$$A_0 = -\int_0^1 Q_w(X)\,\mathrm{d}X$$

$$A_n = \frac{-2}{n\pi}\int_0^1 Q_w(X) \cos \lambda_n X\,\mathrm{d}X \qquad n = 1, 2, \ldots .$$

The constants B_0, B_1, B_2, . . . remain to be determined. From (3), the temperature and the temperature gradient at the solid–liquid interface ($Y = b$) may be determined:

$$\theta_s(X, b) = A_0 b + B_0 + \sum_{n=1}^{\infty} \cos \lambda_n X(A_n \sinh \lambda_n b + B_n \cosh \lambda_n b) \qquad (4)$$

$$\frac{\partial \theta_s(X, b)}{\partial Y} = A_0 + \sum_{n=1}^{\infty} \cos \lambda_n X(A_n \lambda_n \cosh \lambda_n b + B_n \lambda_n \sinh \lambda_n b) \qquad (5)$$

The as yet unknown constants B will be determined by the use of equations (4) and (5), and the differential equation and boundary conditions for the fluid. The Laplace transform of (1) yields

$$Vs\bar{\theta}_f = \frac{\mathrm{d}^2 \bar{\theta}_f}{\mathrm{d}Y^2} \qquad (6)$$

which has a solution

$$\bar{\theta}_f = C \exp\left[-V^{\frac{1}{2}} s^{\frac{1}{2}} Y\right] + D \exp\left[+V^{\frac{1}{2}} s^{\frac{1}{2}} Y\right] \qquad (7)$$

The constant D is zero since the temperature must remain finite at $Y = \infty$. From (7), the gradient of the transformed fluid temperature at $Y = b$ is seen to be

$$\frac{\partial \bar{\theta}_f(s, b)}{\partial Y} = -V^{\frac{1}{2}} s^{\frac{1}{2}} C \exp\left[-V^{\frac{1}{2}} s^{\frac{1}{2}} b\right] \qquad (8)$$

The constant C is evaluated by equating (8) to the transform of (5). If C is found in this manner and then substituted into (7), the transformed temperature distribution in the fluid is seen to be

$$\bar{\theta}_f = -KV^{-\frac{1}{2}}\, s^{-\frac{1}{2}} \exp\left[-V^{\frac{1}{2}}\, s^{\frac{1}{2}}(Y-b)\right] \left\{\frac{A_0}{s} + \sum_{n=1}^{\infty} \frac{s}{s^2+\lambda_n^2}\left[A_n\lambda_n \cosh \lambda_n b + b_n\lambda_n \sinh \lambda_n b\right]\right\} \quad (9)$$

By use of the Convolution theorem the inverse is found to be

$$\left.\begin{aligned}\theta_f = -KV^{-\frac{1}{2}} & \left\{2A_0\left(\frac{X}{\pi}\right)^{\frac{1}{2}} \exp\left[\frac{-V(Y-b)^2}{4X}\right]\right. \\ & -A_0V^{\frac{1}{2}}(Y-b)\,\mathrm{erfc}\left(\frac{V^{\frac{1}{2}}(Y-b)}{2X^{\frac{1}{2}}}\right) \\ & +\sum_{n=1}^{\infty}\left[A_n\lambda_n\cosh\lambda_n b + B_n\lambda_n \sinh \lambda_n b\right] \\ & \left.\int_0^X \cos\lambda_n\eta \frac{\exp\left[\dfrac{-V^{\frac{1}{2}}(Y-b)^2}{4(X-\eta)}\right]}{\pi^{\frac{1}{2}}(X-\eta)^{\frac{1}{2}}}\,\mathrm{d}\eta\right\}\end{aligned}\right\} \quad (10)$$

Equations (3) and (10) now give the temperature distribution in the solid and fluid respectively in terms of the still undetermined constants B_0, B_1, B_2, ... There remains one unused boundary condition to determine these constants, viz. continuity of temperature at $Y = b$.

$$\theta_f(X, b) = \theta_s(X, b) \quad (11)$$

where θ_f and θ_s are given by (10) and (3) respectively. Integrating equation (11) over the length of the plate yields

$$\int_0^1 \theta_f(X, b)\,\mathrm{d}X = \int_0^1 \theta_s(X, b)\,\mathrm{d}X \quad (12)$$

or

$$\left.\begin{aligned} A_0 b + B_0 = -KV^{-\frac{1}{2}} & \left\{\frac{4A_0}{3\pi^{\frac{1}{2}}} + \right. \\ & \sum_{n=1}^{\infty}\left[A_n\lambda_n \cosh \lambda_n b + B_n\lambda_n\sinh\lambda_n b\right] \\ & \left. \times \int_0^1\int_0^X \frac{\cos\lambda_n\eta}{\pi^{\frac{1}{2}}(X-\eta)^{\frac{1}{2}}}\,\mathrm{d}\eta\,\mathrm{d}X\right\}\end{aligned}\right\} \quad (13)$$

Multiplying (11) by $\cos j\pi X$, $j = 1, 2, \ldots$ and integrating yields

$$\int_0^1 \theta_f(X, b) \cos j\pi X\,\mathrm{d}X = \int_0^1 \theta_s(X, b) \cos j\pi X\,\mathrm{d}X \quad j = 1, 2, \ldots \quad (14)$$

or

$$\left.\begin{aligned} \frac{1}{2}\sum_{n=1}^{\infty} & (A_n \sin\lambda_n b + B_n \cosh \lambda_n b) = \\ & -KV^{-\frac{1}{2}}\left\{\frac{2A_0}{\pi^{\frac{1}{2}}}\int_0^1 X\cos j\pi X\,\mathrm{d}X + \right. \\ & \sum_{n=1}^{\infty}\left[A_n\lambda_n\cosh\lambda_n b + B_n\lambda_n\sinh\lambda_n b\right] \\ & \left.\int_0^1 \cos j\pi X \int_0^X \frac{\cos\lambda_n\eta}{\pi^{\frac{1}{2}}(X-\eta)^{\frac{1}{2}}}\,\mathrm{d}\eta\,\mathrm{d}X\right\} \\ & j = 1, 2, \ldots.\end{aligned}\right\} \quad (15)$$

Equations (14) and (15) may be rearranged in the form

$$\sum_{n=0}^{\infty} E_{jn}B_n = F_j \qquad j = 0, 1, 2, \ldots \quad (16)$$

where the E_{jn} and F_j are known. The infinite system of equations was truncated and solved for B_n using Jordan's method. It was found that by using up to a 40×40 matrix that accurate values of B_n could be obtained up to $n = 20$.

Consider now the case of constant distributed heat generation in the solid with no heat input at the outside surface of the plate, i.e. $Q_s = \text{const}$, $Q_w = 0$. The temperature in the solid is seen to be

$$\theta_s(X, Y) = B_o' + \sum_{n=1}^{\infty} B_n' \cos \lambda_n X \cosh \lambda_n Y - \frac{Y^2 Q_s}{2} \quad (17)$$

At $Y = b$, there is

$$\theta_s(X, b) = \frac{-b^2 Q_s}{2} + B_o' + \sum_{n=1}^{\infty} B_n' \cos \lambda_n X \cosh \lambda_n b \quad (18)$$

$$\frac{\partial \theta_s(X, b)}{\partial y} = -bQ_s - \sum_{n=1}^{\infty} B_n' \lambda_n \cos \lambda_n X \sinh \lambda_n b \quad (19)$$

Equations (4) and (5) become for constant Q_w

$$\theta_s(X, b) = -bQ_w + \sum_{n=1}^{\infty} B_n \cos \lambda_n X \cosh \lambda_n b \quad (20)$$

$$\frac{\partial Q_s(X, b)}{\partial Y} = -Q_w + \sum_{n=1}^{\infty} B_n \lambda_n \cos \lambda_n X \sinh \lambda_n b \quad (21)$$

It is seen by comparison of (18) and (19) with (20) and (21) for $Q_w = Q_s b$, that if (18) and (19)

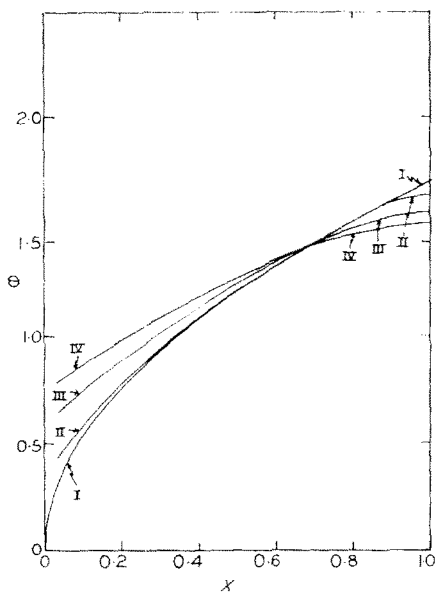

FIG. 2. Temperature at solid–liquid surface. $K = 644$, $V = 1{\cdot}81 \times 10^5$, $Q = 1$. I, $b = 0$; II, $b = 0{\cdot}01$; III, $b = 0{\cdot}05$; IV, $b = 0{\cdot}1$.

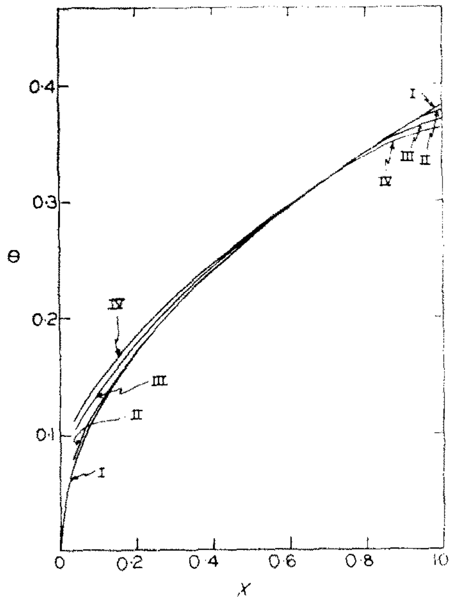

FIG. 3. Temperature at solid–liquid surface. $K = 644$, $V = 3{\cdot}62 \times 10^5$, $Q = 1$. I, $b = 0$; II, $b = 0{\cdot}01$; III, $b = 0{\cdot}05$; IV, $b = 0{\cdot}1$.

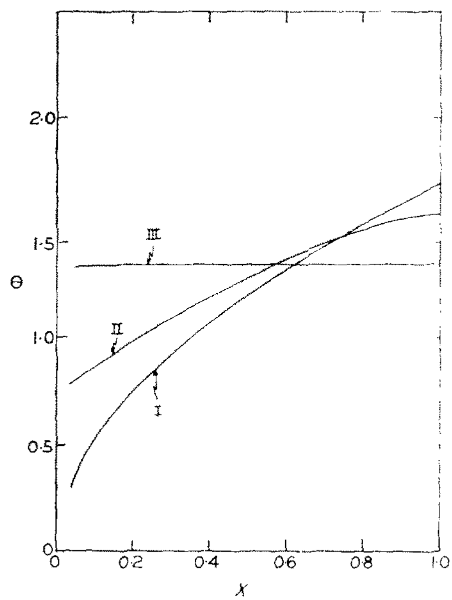

FIG. 4. Temperature at solid–liquid surface. $V = 1{\cdot}81 \times 10^5$, $Q = 644/K$, $b = 0{\cdot}1$. I, $K = 6{\cdot}44$; II, $K = 644$; III, $K = 64\,000$.

were used along with the energy equation for the fluid to determine the constants B'_n, the results would be

$$B'_n = B_n$$

$$B'_o = B_0 - \frac{b^2 Q_s}{2} \tag{22}$$

The temperature at the solid–liquid interface, and throughout the fluid, is therefore the same for the case of constant heat generation throughout the solid as for a constant heat input at the outside surface of the plate. The interface temperature is again given by Figs. 2, 3, and 4. The temperature distribution in the solid is, of course, different in the two cases.

DISCUSSION OF RESULTS

The coefficients B_n and the temperature at the solid–liquid interface were evaluated for $0{\cdot}0 \leqslant b \leqslant 0{\cdot}1$, $6{\cdot}44 \leqslant K \leqslant 64\,000$, and $1{\cdot}81 \times 10^5 \leqslant V \leqslant 1{\cdot}36 \times 10^7$. The effect of wall conduction on heat transfer to the fluid increased with increasing plate thickness to length ratio b, increasing conductivity ratio K, and with decreasing dimensionless velocity V, where V is the product of the Prandtl number and the Reynolds number based on plate length. For $K = 6{\cdot}44$, wall conduction has a negligible effect, even for the lowest fluid velocity and thickest plate considered. The temperature of the solid–fluid interface is shown in Figs. 2 and 3 for $K = 644$, $V = 1{\cdot}81 \times 10^5$ and $3{\cdot}62 \times 10^6$ respectively, with dimensionless plate thickness b as a parameter. For zero plate thickness the temperature assumes the well-known parabolic shape, as shown by the curves $b = 0$. For increasing b, the temperature distribution tends to be leveled off, as would be expected. From Fig. 2, it is seen that for $V = 1{\cdot}81 \times 10^5$ there is an effect of wall conduction even for the thinnest plate considered ($b = 0{\cdot}01$). The effect increases with increasing plate thickness. For $V = 3{\cdot}62 \times 10^6$ (Fig. 3), the effect is very small for $b = 0{\cdot}01$. Figure 4 shows the surface temperature distribution as a function of K for a plate thickness of 0·1. $K = 6{\cdot}44$ gives a surface temperature distribution which is essentially that which would be obtained by neglecting the plate conduction. The highest conductivity ratio considered, $K = 64\,400$, results in a surface temperature distribution which is almost constant; a constant temperature would result from an infinite K. Values for the coefficients B_n as well as the temperature at the liquid–solid surface for other values of the parameters can be found in reference [5].

In summary, it may be said that for the physical situation considered here that wall conduction will not be important for $b \leqslant 0{\cdot}01$ or for $K \leqslant 6{\cdot}44$. If both of these conditions are not met, wall conduction may or may not be important depending on the values of the other parameters. This is to be contrasted to the results on an infinite length plate of finite thickness [1] in which even in the case of $K = 0{\cdot}4$ the surface temperature distribution is affected by wall conduction because of the upstream conduction. A study of the effect of wall conduction on peripheral temperature distributions [4] in a channel showed that there is an effect of wall conduction for lower values of K than would be significant with the geometry considered in this work.

It is of interest to consider the effect of plate length on the heat transfer. For small values of K, plate length has a negligible influence. When K is small, the solution approaches that of heat transfer to a slug flow neglecting wall conduction, in which case plate length is not important. For higher values of K, increasing the plate length will increase the solid–liquid interface temperature and therefore the heat transfer to the liquid at every position along the plate if the heat per unit length introduced to the system is the same. This may be most easily seen for very large K. In this case the interface temperature approaches a constant. In order for the total heat transferred to the liquid to remain equal to the total heat introduced to the plate, the heat transfer to the liquid at an upstream position must increase to compensate for the fact that at a downstream position the heat flux to the fluid will be less than that into the plate at the same value of X.

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Résumé—Le problème du transport de chaleur avec un écoulement à vitesse uniforme le long d'une plaque plane de dimensions finies est étudié. La chaleur est introduite dans le système, soit au moyen d'un flux de chaleur connu à la surface extérieure de la plaque, soit au moyen d'une source de chaleur répartie dans la plaque. Une conduction bidimensionnelle se produit dans la plaque. La distribution de température dans la plaque et le fluide est obtenue sous la forme d'un développement en série.

Zusammenfassung—Das Problem des Wärmeübergangs bei einer Kolbenströmung entlang einer ebenen Platte endlicher Abmessungen wird theoretisch untersucht. Wärme wird dem System entweder als bekannter Wärmefluss an der äusseren Plattenoberfläche oder durch verteilte Wärmequellen in der Platte zugeführt. Zweidimensionale Leitung tritt in der Platte auf. Die Temperaturverteilung in der Platte und in der Flüssigkeit ist in Form einer Reihe angegeben.

Аннотация—Рассмотрена двумерная задача о теплообмене при обтекании плоской пластины конечных размеров при ползущем режиме течения. Тепло подводится к системе либо заданным тепловым потоком на внешней поверхности, либо при помощи распределенного источника тепла в пластине. Распределение температуры в пластине и в жидкости получено в виде ряда.